THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE two-hundredth regular meeting of the Society was held in New York City on Saturday, October 26, 1918. War conditions and the prevailing epidemic cooperated to reduce the attendance, which included the following eleven members:

Professor E. W. Brown, Professor F. N. Cole, Professor T. S. Fiske, Mr. T. C. Fry, Professor O. E. Glenn, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Professor P. H. Linehan, Professor R. L. Moore, Professor H. W. Reddick.

Professor E. W. Brown presided at the morning session, and Professor O. E. Glenn at the afternoon session. The Council announced the election of the following persons to membership in the Society: Professor R. A. Arms, Juniata College; Professor M. D. Earle, Furman University; Professor Ernest Flammer, Queen's University; Professor Gillie A. Larew, Randolph-Macon Woman's College; Dr. Flora E. Le Stourgeon, Mt. Holyoke College; Professor John Matheson, Queen's University; Dr. F. R. Morris, University of California; Professor Susan M. Rambo, Smith College; Dr. W. G. Simon, Adelbert College. One application for membership in the Society was received.

A committee was appointed to audit the accounts of the Treasurer for the current year. A list of nominations for officers and other members of the Council was adopted and ordered printed on the official ballot for the annual election.

A committee was also appointed to collect subscriptions from members of the Society and others for the purpose of establishing at Harvard University a suitable memorial of the late Professor Maxime Bôcher, President of the Society in 1909–1910.

The Chicago meeting in the Christmas holidays was designated by the Council as the annual meeting of the Society for this year, the usual eastern meeting being cancelled. Members attending the Baltimore meeting of the American Association were invited to read papers before Section A, having previously registered titles and abstracts with the Secretary for record in the report of the annual meeting.

The Southwestern Section decided not to hold its usual meeting this year. The February, 1919, meeting of the Society will also be omitted. The usual meetings will be held in April.

The following papers were read at the October meeting:

(1) Dr. J. E. McAtee: "The transformation of a regular group into its conjoint."

(2) Professor E. W. Chittenden: "On the Heine-Borel property in the theory of abstract sets."

(3) Professor Gillie A. Larew: "Necessary conditions for the problem of Mayer in the calculus of variations."

(4) Professor D. M. Y. Somerville: "Quadratic systems of circles in non-euclidean geometry."

(5) Professor M. B. Porter: "Derivativeless continuous functions."

(6) Dr. G. H. Hallett, Jr.: "Concerning the definition of a simple continuous arc."

(7) Professor R. L. Moore: "A characterization of Jordan regions by properties having no reference to their boundaries."

(8) Professor R. L. Moore: "Concerning simple continuous curves."

(9) Professor Edward Kasner: "Fields of force and Monge equations."

Dr. Hallett's paper was communicated to the Society through Professor R. L. Moore. In the absence of the authors the papers of Dr. McAtee, Professor Chittenden, Professor Larew, Professor Somerville, Professor Porter, and Dr. Hallett were read by title. Abstracts of the papers follow below.

1. The statement that there is a transformation of order 2 that transforms a regular substitution group into its conjoint occurs in the literature. In this paper Dr. McAtee exhibits such a transformation.

2. Professor Chittenden's paper appeared in full in the November Bulletin.

3. Since A. Mayer formulated that problem of the calculus of variations usually called by his name, considerable attention has been given to the deduction of the Euler-Lagrange equations of the problem as a first necessary condition. Little attention, subsequent to the original paper of Mayer, has been

paid to the necessary conditions analogous to those of Legendre and Jacobi or to the deduction of a necessary condition analogous to that of Weierstrass. It is the purpose of Professor Larew's paper to inquire systematically into the question of necessary conditions. In the investigation of the corner-point condition for the so-called "discontinuous solutions," the theorem on the necessary condition of Euler is extended to include arcs which are continuous, but which may have a finite number of corners. A formulation and proof is supplied for the necessary condition of Weierstrass, and the Legendre condition follows directly. The Jacobi condition is deduced in much more simple fashion than usual by an application of the Euler equations and the corner-point condition to the second variation.

4. Professor Somerville's paper discusses the quadratic systems of circles with one or two parameters in non-euclidean geometry. The general one-parameter system consists of one of three systems of circles all having double contact with a fixed conic K. The limiting points are a pair of foci, and the limiting lines are a pair of focal lines of K. Coaxal and homocentric systems are degenerate cases, and a third degenerate case consists of a system of equal circles. The two-parameter quadratic system is found to be composed of one-parameter systems whose conic envelopes all have double contact with a fixed conic. A specialized system is noteworthy in which every circle cuts a fixed circle orthogonally; another system, reciprocal to this, has the property that every circle is tangentially distant a quadrant from a fixed circle

5. Professor Porter's paper deals with continuous functions which either have nowhere a derivative or only possess derivatives inside an interval or at a point. A simple method is developed for constructing such functions in a large variety and it is shown that in the case of Weierstrass's type the condition that a be an odd integer is unnecessary.

6. In a paper entitled "Curves in non-metrical analysis situs with an application in the calculus of variations," *American Journal of Mathematics*, volume 33 (1911), pages 285–326, N. J. Lennes gives the following definition of a simple continuous arc. "A continuous simple arc connecting two points

A and B, $A \neq B$, is a bounded, closed, connected set of points $[A]$ containing A and B such that no connected proper subset of $[A]$ contains A and B." The purpose of Dr. Hallett's paper is to show that the word "bounded" in this definition is superfluous.

7. Schoenflies* has formulated a set of conditions under which the common boundary of two domains will be a simple closed curve. A different set has been given by J. R. Kline (cf. this BULLETIN, July, 1918, page 471). Carathéodory† has secured conditions under which the boundary of a single domain will be such a curve. In each of these treatments conditions are imposed (1) on the boundary itself, (2) regarding the relation of the boundary to the domain or domains in question. In the present paper Professor Moore establishes the following theorem in which all the conditions imposed are on the domain R alone:

In order that a simply connected limited two-dimensional domain R should have a simple closed curve as its boundary it is necessary and sufficient that R should be uniformly connected im kleinen.

A point set M is said to be uniformly connected im kleinen‡ if for every positive number e there exists a positive number δ_e such that if P_1 and P_2 are two points of M at a distance apart less than δ_e they lie together in a connected subset of M every point of which is at a distance less than e from P_1.

8. Various definitions have been given of simple continuous arcs, simple closed curves and open continuous curves. In the present paper Professor Moore defines these terms from a point of view which, as far as he knows, is different from any hitherto employed for this purpose. He makes use of the notion of the boundary of a point set with respect to a point set that contains it. If M is a proper subset of N, a boundary point of M with respect to N is a point which is a point or a limit point of M and also a point or a limit point of $N - M$.

Definition 1. A simple continuous closed curve is a continuous§ point set M every continuous proper subset of which has two and only two boundary points with respect to M.

* *Göttinger Nachrichten*, 1902, p. 185.

† *Math. Annalen*, vol. 73 (1912–13), p. 366.

‡ Cf. H. Hahn, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 23 (1914), pp. 318–322.

§ A set of points is said to be continuous if it is closed and connected.

Definition 2. A simple continuous open curve is a continuous point set M every continuous bounded subset of which has just two boundary points with respect to M.

Definition 3. If A and B are two distinct points, a simple continuous arc from A to B is a continuous point set M containing A and B such that (1) no continuous subset of M has more than two boundary points with respect to M, (2) a subset K of M has just two boundary points with respect to M if and only if K contains neither A nor B.

He shows that these point sets may also be defined in the following simple manner.

Definition 1′. A simple continuous closed curve is a continuous bounded point set which is disconnected by the omission of any two of its points.

Definition 2′. A simple continuous open curve is a continuous point set which is disconnected by the omission of any one of its points.

Definition 3′. If A and B are two distinct points, a simple continuous arc from A to B is a continuous bounded point set which contains A and B and is disconnected by the omission of any one of its points other than A or B.

Every simple closed curve as defined in 1 or 1′ is in one-to-one continuous correspondence with a circle. Every simple continuous open curve as defined in 2 or 2′ is in one-to-one continuous correspondence with a straight line and every simple continuous arc as defined in 3 or 3′ is in one-to-one continuous correspondence with a linear interval. As has been shown by Dr. G. H. Hallett, Jr., in Lennes's definition of a simple continuous arc the condition that the point set in question should be *bounded* is superfluous. In Definition 3′ this is not the case.

9. After observing that the ∞^5 trajectories defined by a general field of force in space cannot all satisfy a Monge equation, Professor Kasner raises the question under what conditions this is possible for ∞^4 of the trajectories. It is shown that the Monge equation must be linear (that is, Pfaffian) and that it must define a null system. The direction of the force vector at any point must be in the null system. All forces of this type are determined explicitly.

F. N. Cole,
Secretary.

ON THE EVALUATION OF THE ELLIPTIC TRANSCENDENTS η_2 AND η_2'.

BY PROFESSOR HARRIS HANCOCK.

WITH Halphen* write

$$f(s) = 4s^3 - g_2 s - g_3 = 4(s - e_1)(s - e_2)(s - e_3) = S,$$

where e_2 is a real quantity and e_1, e_3 are conjugate imaginaries such that $e_1 - e_3$ is a positive pure imaginary. The discriminant $\Delta = g_2^3 - 27g_3^2$ is here negative. The elliptic differential equation in this notation is $ds^2/du^2 = 4s^3 - g_2 s - g_3$, whose integral is

$$u = \int_s^\infty \frac{ds}{\sqrt{f(s)}}.$$

If the lower limit of this integral be considered as a function of the integral, that is of u, we may write $s = \wp u$.

As is well known, the quantity ω_2 is defined by

$$(1) \qquad \omega_2 = \int_{e_2}^\infty \frac{ds}{\sqrt{f(s)}}; \quad \wp\omega_2 = e_2, \quad \wp'\omega_2 = 0.$$

If further we write

$$f_1(s) = 4s^3 - g_2 s + g_3 = 4(s + e_1)(s + e_2)(s + e_3),$$

the quantity ω_2' is defined through the relations

$$(2) \qquad \frac{\omega_2'}{i} = \int_{-e_2}^\infty \frac{ds}{\sqrt{f_1(s)}}; \quad \wp\omega_2' = e_2, \quad \wp'\omega_2' = 0.$$

Note that ω_2' is a pure imaginary.

It may be shown that the three half-periods

$$\omega_1 = \tfrac{1}{2}(\omega_2 - \omega_2'), \quad \omega_2, \quad \omega_3 = \tfrac{1}{2}(\omega_2 + \omega_2'),$$

correspond to the quantities e_1, e_2, e_3 above, the relations being the same as are those for ω, $\omega'' = \omega + \omega'$, ω' in the usual notation where all three of the roots e_1, e_2, e_3 are real.

Write

$$\frac{d}{du}\zeta u = -\wp u,$$

* Fonctions Elliptiques, vol. 1, chap. 6.

note that $\zeta(-u) = -\zeta u$ and that the quantities η_1 and η_3 are defined through the relations $\zeta\omega_1 = \eta_1$ and $\zeta\omega_3 = \eta_3$. Further write

$$\eta_1 = \tfrac{1}{2}(\eta_2 - \eta_2'), \quad \eta_3 = \tfrac{1}{2}(\eta_2 + \eta_2').$$

In practically all the text books on this subject it is shown that the two half-periods ω_1 and ω_2 play with respect to the function ζu the same rôle, when the discriminant is negative, as is played by the half-periods ω and ω' when the discriminant is positive; and the constants η_1 and η_3 which correspond to these half-periods are connected with η_2 and η_2' by the same relations which connect ω_1, ω_3 with ω_2, ω_2'. The values of η_2 and η_2' may be derived as follows:

In my monograph on Elliptic Integrals, page 51, formulas (39) and (40), it is shown that

$$\int_a^x \frac{dx}{\sqrt{(x-\alpha)[(x-\rho)^2+\sigma^2]}} = \frac{1}{\sqrt{M}} \operatorname{cn}^{-1}\left[\frac{M-(x-\alpha)}{M+(x-\alpha)}, k\right], \tag{3}$$

where

$$k^2 = \frac{1}{2} - \frac{1}{2}\frac{\alpha-\rho}{M}, \quad M^2 = (\rho-\alpha)^2 + \sigma^2;$$

and

$$\int_x^a \frac{dx}{\sqrt{(\alpha-x)[(x-\rho)^2+\sigma^2]}} = \frac{1}{\sqrt{M}} \operatorname{cn}^{-1}\left[\frac{M-(\alpha-x)}{M+(\alpha-x)}, k'\right], \tag{4}$$

where

$$k'^2 = \frac{1}{2} + \frac{1}{2}\frac{\alpha-\rho}{M}; \quad k^2 + k'^2 = 1, \quad 2kk' = \frac{\sigma}{M}.$$

These two formulas may be made the foundation of the whole so-called Weierstrassian theory, with negative discriminant.

To save the reader the trouble of deriving them and simply as a verification, write

$$u = \operatorname{cn}^{-1}\left[\frac{M-(x-\alpha)}{M+(x-\alpha)}\right]$$

or

$$\operatorname{cn} u = \frac{M-(x-\alpha)}{M+(x-\alpha)} = \cos\varphi, \text{ where } \varphi \text{ is put} = \operatorname{am} u.$$

It follows at once that

$$x-\alpha = M\frac{1-\cos\varphi}{1+\cos\varphi}, \quad dx = 2M\frac{\sin\varphi\, dx}{(1+\cos\varphi)^2},$$

$$(x-\rho)^2+\sigma^2 = \frac{2M^2(1+\cos^2\varphi)+2(\alpha-\rho)M\sin^2\varphi}{(1+\cos\varphi)^2}.$$

These values substituted in (3) cause that integral to become

$$\frac{1}{\sqrt{M}}\int_0^{\phi}\frac{dx}{\sqrt{1-k^2\sin^2\varphi}}, \quad \text{where} \quad k^2 = \frac{1}{2}-\frac{1}{2}\frac{\alpha-\rho}{M}.$$

The indicated transformation reduces the integral (4) to the corresponding Legendre normal form.

Note that when $x=\alpha$, then $\varphi = \frac{1}{2}\pi$. From (3) and (4) it follows at once that

$$\omega_2 = \frac{K}{\sqrt{M}}, \quad \omega_2' = \frac{iK'}{\sqrt{M}}, \tag{5}$$

where

$$K = \int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2\varphi}}, \quad K' = \int_0^{\pi/2}\frac{dx}{\sqrt{1-k'^2\sin^2\varphi}},$$

k^2 and k'^2 having the values already indicated.

If one wished to evaluate an elliptic integral of the second kind in which there appeared a denominator, as in the integrals (3) and (4), similar reductions would transform it into the corresponding Legendre normal form and its value could be taken at once from the Legendre tables.

If however the Weierstrassian formulas are introduced, we may proceed as follows: From the definition above

$$\zeta u = -\int \wp u\, du;$$

or, since $\wp u = s$ and

$$du = \frac{ds}{\sqrt{4s^3-g_2 s-g_3}},$$

we may write*

$$\zeta u = \int^{s} \frac{sds}{\sqrt{4s^3 - g_2 s - g_3}},$$

with suitable choice of the constant in the lower limit.

If we put $e_1 = \rho + i\sigma$, $e_3 = \rho - i\sigma$, it is seen that

$$\int_{e_1}^{e_3} \frac{sds}{\sqrt{S}} = \int_{\rho+i\sigma}^{\rho-i\sigma} \frac{sds}{2\sqrt{(s-e_2)(s-e_1)(s-e_3)}}$$

$$= \int_{\rho+i\sigma}^{\rho-i\sigma} \frac{sds}{2\sqrt{(s-e_2)[(s-\rho)^2+\sigma^2]}}.$$

Observe that the left-hand integral is

$$\zeta\omega_3 - \zeta\omega_1 = \tfrac{1}{2}(\eta_2 + \eta_2') - \tfrac{1}{2}(\eta_2 - \eta_2') = \eta_2'.$$

In the right-hand integral put $s = \rho - it$ and it follows that

$$-\eta_2' = \int_{t=\sigma}^{t=-\sigma} \frac{(\rho - it)idt}{2\sqrt{(3\rho - it)(t+\sigma)(\sigma - t)}}. \tag{6}$$

To evaluate this integral write with Lucien Lévy† (see page 95)

$$\frac{1}{\sqrt{3\rho - it}} = \alpha + i\beta.$$

We have at once

$$\frac{3\rho + it}{9\rho^2 + t^2} = \alpha^2 - \beta^2 + i2\alpha\beta,$$

or

$$\alpha^2 - \beta^2 = \frac{3\rho}{9\rho^2 + t^2}, \quad 2\alpha\beta = \frac{t}{9\rho^2 + t^2},$$

$$\alpha^2 + \beta^2 = \frac{1}{9\rho^2 + t^2}.$$

As the sign of α is arbitrary, we shall take it positive; while the sign of β, from the second of the formulas just written, is the same as the sign of t. This may be indicated by the notation ϵ_t.

*See for example Schwarz, Formeln, etc., p. 87.

†Précis élémentaire de la Théorie des Fonctions Elliptiques, published by Gauthier-Villars, Paris, 1898.

It follows at once that

$$\alpha = \frac{1}{\sqrt{2}} \frac{\sqrt{3\rho + \sqrt{9\rho^2 + t^2}}}{\sqrt{9\rho^2 + t^2}},$$

$$\beta = \epsilon_t \frac{\sqrt{\sqrt{9\rho^2 + t^2} - 3\rho}}{\sqrt{9\rho^2 + t^2}}.$$

Write $(\rho - it)i(\alpha + i\beta)$ in the form $\alpha t - \beta\rho + i(\alpha\rho + \beta t)$ and put $\eta_2' = A + iB$. From formula (6) it is seen that

$$-A = \int_{t=-\sigma}^{t=+\sigma} \frac{dt}{2\sqrt{2}} \frac{1}{\sqrt{9\rho^2 + t^2}\sqrt{\sigma^2 - t^2}} \times \left[t\sqrt{3\rho + \sqrt{9\rho^2 + t^2}} - \rho\epsilon_t \sqrt{\sqrt{9\rho^2 + t^2} - 3\rho} \right],$$

$$-B = \int_{t=-\sigma}^{t=+\sigma} \frac{dt}{2\sqrt{2}} \frac{1}{\sqrt{9\rho^2 + t^2}\sqrt{\sigma^2 - t^2}} \times \left[\rho\sqrt{3\rho + \sqrt{9\rho^2 + t^2}} + \epsilon_t \sqrt{\sqrt{9\rho^2 + t^2} - 3\rho} \right].$$

If we write A in the form

$$\int_{-\sigma}^{0} + \int_{0}^{\sigma}$$

and in either of these integrals make the substitutions $t = -\tau$, it is seen that the one is the negative of the other, so that $A = 0$. This was evident à priori, since $\eta_2' = \zeta(\omega_2')$ is a pure imaginary, ζ being an odd function and ω_2' a pure imaginary.

It also follows that

$$B = \frac{\eta_2'}{i} = -\int_{t=0}^{t=\sigma} \frac{dt}{\sqrt{2}\sqrt{9\rho^2 + t^2}} \frac{1}{\sqrt{\sigma^2 - t^2}} \times \left[\rho\sqrt{3\rho + \sqrt{9\rho^2 + t^2}} + t\sqrt{\sqrt{9\rho^2 + t^2} - 3\rho} \right].$$

In this expression put $9\rho^2 + t^2 = x^2$, and it becomes, if $t > 0$,

$$\frac{\eta_2'}{i} = -\int_{x=3\rho}^{x=M} \frac{(x - 2\rho)dx}{\sqrt{2(x - 3\rho)(M^2 - x^2)}}, \tag{7}$$

where

$$M^2 = 9\rho^2 + \sigma^2.$$

To evaluate the integral (7), write $x = -t + \rho$, and we have

$$(8) \qquad \frac{-\eta_2'}{i} = \sqrt{2}\int_{t=-2\rho}^{t=\rho-M} \frac{(t+\rho)dt}{2\sqrt{(t-t_1)(t-t_2)(t-t_3)}},$$

where

$$t_1 = \rho + M, \quad t_2 = -2\rho, \quad t_3 = \rho - M; \quad t_1 + t_2 + t_3 = 0.$$

Write

$$u = \int_t^\infty \frac{dt}{\sqrt{4(t-t_1)(t-t_2)(t-t_3)}},$$

so that $t = \wp u$, an elliptic function with half-periods, say, $\omega, \omega', \omega'' = \omega + \omega'$.

The above transformations, although somewhat complicated, are instructive and are due to Lévy, loc. cit., pages 96 and 97. In the next substitution, however, he makes a rather curious mistake: For du he writes the value just given and associated with it he writes $t = \wp(u + \omega'')$. This of course vitiates his results, which are repeated under the heading "Principal Formulas," page 221. The formula (8) becomes*

$$\frac{-\eta_2'}{i} = \sqrt{2}\int_{\omega''}^{\omega'} (\wp u + \rho)du$$

$$= \sqrt{2}[-\zeta\omega' + \zeta\omega'' + \rho\omega' - \rho\omega''] = \sqrt{2}(\eta - \rho\omega).$$

The values of η and η' expressed in the Legendre normal forms are given in most text-books; see for example Schwarz, loc. cit., page 34. Schwarz fails, however, to give the corresponding values of η_2 and η_2'.

It is seen that

$$\frac{-\eta_2'}{i} = \sqrt{2}\left[\sqrt{t_1 - t_3}\left(E' - \frac{t_1 K'}{t_1 - t_3}\right) - \rho\frac{K'}{\sqrt{t_1 - t_3}}\right]$$

$$= \sqrt{M}\left[2E' - \frac{M + 2\rho}{M}K'\right].$$

* Here again we must be careful regarding the sign of $dt/du = +\sqrt{\ }$. See, for example, Appell et Lacour, Fonctions Elliptiques, p. 72.

The integral in formula (8) may be again calculated as follows: Write*

$$\sin^2\varphi = \frac{M-x}{M-3\rho},$$

so that

$$M - x = (M-3\rho)\sin^2\varphi,$$

$$x - 3\rho = (M-3\rho)\cos^2\varphi, \quad M + x = 2M[1 - k'^2\sin^2\varphi],$$

where

$$k'^2 = \frac{1}{2} - \frac{3\rho}{2M}, \quad dx = (3\rho - M)2\sin\varphi\cos\varphi\, d\varphi.$$

The integral in question becomes

$$\frac{1}{\sqrt{M}}\int_0^{\pi/2} \frac{(3\rho - M)\sin^2\varphi + M - 2\rho}{\sqrt{1 - k'^2\sin^2\varphi}}\, d\varphi$$

$$= \frac{2M}{\sqrt{M}}\int_0^{\pi/2} \frac{1 - k'^2\sin^2\varphi}{\sqrt{1 - k'^2\sin^2\varphi}}$$

$$- \frac{2M}{\sqrt{M}}\int^{\pi/2} \frac{d\varphi}{\sqrt{1 - k'^2\sin^2\varphi}} + \frac{M - 2\rho}{\sqrt{M}}K'$$

$$= 2\sqrt{M}E' - \frac{M + 2\rho}{\sqrt{M}}K', \text{ as above.}$$

Note that $e_2 = -2\rho$. It is seen at once from formulas (1) and (2) above that η_2' is had from η_2 by changing the sign of e_2 and multiplying the value of η_2 by $-i$ and putting k' for k. We thus have

$$\eta_2 = \sqrt{M}\left[2E - \left(1 - \frac{2\rho}{M}\right)K\right], \quad 2\rho = -e_2, \quad M^2 = 9\rho^2 + \sigma^2,$$

$$\eta_2' = -i\sqrt{M}\left[2E' - \left(1 + \frac{2\rho}{M}\right)K'\right],$$

$$\eta_2\omega_2' - \eta_2'\omega_2 = i[2EK' + 2E'K - 2KK] = i2\pi.$$

The relation to the right is, of course, Legendre's celebrated formula.

* See formula (17), page 45, of my monograph mentioned above.

One can judge from the present paper the little value of the Weierstrassian formulas when the applications of the general theory are involved or whenever any kind of numerical computation is desired. One sees also how easy it is to introduce errors in the calculation. The book of Lévy already mentioned is in most respects an excellent work, certainly from the standpoint of applied mathematics, much of the material being new; but when a substitution involving an elliptic function in the Weierstrassian form is introduced, the book is not free from errors. For example, not to mention many inaccuracies, besides the mistake already cited, it is seen that on page 82 of Lévy's book $e_1 + e_2 + e_3 \neq 0$. The same error is found on page 156, while in the calculation of ζu, the functions introduced on page 104 are incorrectly given. At the end of my larger work, volume 1, the Weierstrassian functions are put in juxtaposition with those of the older theory and it is seen that thereby nothing new is added.

UNIVERSITY OF CINCINNATI,
June 14, 1918.

ON PLANE ALGEBRAIC CURVES WITH A GIVEN SYSTEM OF FOCI.

BY PROFESSOR ARNOLD EMCH.

(Read before the American Mathematical Society, April 27, 1918.)

1. Let

$$\phi(u, v, w) = 0 \tag{1}$$

be a curve of class n, and

$$u\xi + v\eta + w\zeta = 0 \tag{2}$$

a line with the coordinates u, v, w, and $x' = \xi/\zeta, y' = \eta/\zeta$ current cartesian coordinates. Then

$$-1 \cdot \xi - i \cdot \eta + (x + iy)\zeta = 0 \tag{3}$$

is a line which passes through the point (x, y) and the circular point I with the slope i. The coordinates of (3) are $u = -1$,

$v = -i, w = x + iy$. Consequently, if these satisfy (1), then the line (3) will be a tangent of (1); and as it passes through the point (x, y), this point will be a focus of the curve. Hence, to find the real foci of (1), we have to determine real values of x and y in such a manner that

$$\phi(-1, -i, x + iy) = 0. \tag{4}$$

In other words, we have to solve the equation of the nth degree in w,

$$\phi(-1, -i, w) = 0. \tag{5}$$

The n complex roots of this equation interpreted in the cartesian plane are the required foci.* These are also obtained by writing (4) in the form $f(x, y) + ig(x, y) = 0$, where f and g are real polynomials in x and y, and by finding the intersections of f and g. In this manner we get n^2 solutions of which n are real and correspond to the n roots of (5). The remaining $n(n-1)$ imaginary solutions are the *associates* of the n roots of (5).

2. Conversely, when the foci $z_1, z_2, \cdots, z_n$ of a curve are given in a complex plane, we can easily form the equation

$$\prod_{k=1}^{n}(z - z_k) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n = 0, \tag{6}$$

with these as roots. The curve of the nth class with these as foci has the form

$$\begin{aligned}\phi \equiv\ & w^n + (\alpha_{11}v + \alpha_{12}u)w^{n-1} + (\alpha_{21}v^2 + \alpha_{22}vu + \alpha_{23}u^2)w^{n-2} \\ & + (\alpha_{31}v^3 + \alpha_{32}v^2u + \alpha_{33}vu^2 + \alpha_{34}u^3)w^{n-3} + \cdots \\ & + \alpha_{n1}v^n + \alpha_{n2}v^{n-1}u + \alpha_{n3}v^{n-2}u^2 + \ldots + \alpha_{nn}u^n = 0.\end{aligned} \tag{7}$$

The foci are conversely obtained by putting in this equation $u = -1, v = -i$, and solving for w. In this manner from (7) we get

$$\begin{aligned}& w^n + (i\alpha_{11} - \alpha_{12})w^{n-1} + (-\alpha_{21} + i\alpha_{22} + \alpha_{23})w^{n-2} \\ & \quad + (i\alpha_{31} + \alpha_{32} - i\alpha_{33} - \alpha_{34})w^{n-3} + \cdots \\ & \quad + (-i)^n\alpha_{n1} - (-i)^{n-1}\alpha_{n2} + (-i)^{n-2}\alpha_{n3} + \cdots \\ & \qquad + (-1)^n\alpha_{nn} = 0.\end{aligned} \tag{8}$$

* Siebeck: "Ueber eine neue analytische Behandlungsweise der Brennpunkte," *Crelle's Journal*, vol. 64, pp. 175–182 (1865).

In this equation we can determine real values of the α_{jk} $(j, k = 1, 2, 3, \cdots, n)$ in an infinite number of ways in such a manner that (8) becomes identical with (6). For any such a set of α_{jk}'s (7) represents a curve of the nth class with the given points as foci.

To obtain the coordinates of (7) in projective point coordinates ξ, η, ζ, we must eliminate w, v, u between (7) and the three equations

$$\xi = \frac{\partial \phi}{\partial u}, \qquad \eta = \frac{\partial \phi}{\partial v}, \qquad \zeta = \frac{\partial \phi}{\partial w}, \tag{9}$$

as is well known.

3. When an n-ic $F(x, y) = 0$ is given, the foci may be found by making the substitution $y = ix + c$, and by establishing the discriminant for the equation $F(x, ix + c) = 0$ in x. This leads to an equation in c whose roots determine the foci. If $c_k = A_k + iB_k$ is such a root, then, as $c = y - ix$, the coordinates of the corresponding focus are $x = -B_k$, $y = A_k$.

Another method for finding the foci of an n-ic is suggested by the foregoing results: Establish the equation $\phi(u, v, w) = 0$ corresponding to $F(x, y) = 0$ by the well known method of elimination, as suggested above for the converse case.

4. As an example consider the circular cubic*

$$F \equiv (x^2 + y^2)x + yz^2 = 0, \tag{10}$$

whose cartesian equation is obtained by putting $z = 1$. The equation of the equivalent curve of class (6), in line coordinates u, v, w is obtained by eliminating x, y, z between (10) and

$$u = \partial F/\partial x = 3x^2 + y^2, \qquad v = 2xy + z^2, \qquad w = 2yz. \tag{11}$$

After this rather tedious process the required equation becomes

$$4w^6 + 24uvw^4 - (u^4 - 30u^2v^2 - 27v^4)w^2 - 4u^3v(u^2 + v^2) = 0. \tag{12}$$

Setting $u = -1, v = -i$, this reduces to

$$w^6 + 6iw^4 - w^2 = 0, \tag{13}$$

* See *Annals of Mathematics*, vol. 14, pp. 57–71 (1912).

whose roots are $w_{1,2,3,4} = \pm \frac{1}{2}\sqrt{2}(1+i)(-3 \pm \sqrt{8})$, $w_5 = 0, w_6 = 0$. The coordinates of the foci F_i of the circular cubic (10) are therefore

$$F_1(x_1 = y_1 = 2 - \tfrac{3}{2}\sqrt{2}), \quad F_2(x_2 = y_2 = -2 + \tfrac{3}{2}\sqrt{2}),$$

$$F_3(x_3 = y_3 = -2 - \tfrac{3}{2}\sqrt{2}), \quad F_4(x_4 = y_4 = 2 + \tfrac{3}{2}\sqrt{2}),$$

$$F_5(x_5 = y_5 = 0), \quad F_6(x_6 = y_6 = 0).$$

F_5 and F_6 coincide with the origin, and form a quadruple focus which lies on the curve. It is called quadruple, since it may be considered as the intersection of two two-fold tangents at the isotropic points.

The foci F_1, F_2, F_3, F_4 of a circular cubic lie on a circle, which in this case degenerates into a straight line, the bisecting line of the first and third quadrant.

5. *Confocal conics.*—Let $F_1(1, 0)$ and $F_2(-1, 0)$ be the foci of a system of confocal conics. Then the determining equation is $w^2 - 1 = 0$, and the system of curves of the second class with these foci is

$$w^2 - \alpha u^2 - (\alpha - 1)v^2 = 0.$$

The corresponding cartesian equation is

$$\frac{x^2}{\alpha} + \frac{y^2}{\alpha - 1} = 1,$$

which, for a variable α, represents a system of confocal conics.

6. *A certain class of curves, admitting the nth roots of unity as foci.* The equation determining these foci is

$$w^n - 1 = 0, \tag{14}$$

and the corresponding curve of class n has the form

$$w^n - (\alpha v^n + \alpha_1 v^{n-1}u + \alpha_2 v^{n-2}u^2 + \alpha_3 v^{n-3}u^3 + \cdots + \alpha_n u^n) = 0, \tag{15}$$

in which, when n is even, say $n = 2\lambda$,

$$\alpha(-1)^\lambda + \alpha_2(-1)^{\lambda-1} + \alpha_4(-1)^{\lambda-2} + \cdots = 1.$$

For an even λ, this becomes

$$\alpha - \alpha_2 + \alpha_4 - \cdots = 1.$$

For any λ, there is $\alpha_1 - \alpha_3 + \alpha_5 - \cdots = 0$. When λ is odd, then $-\alpha + \alpha_2 - \alpha_4 + \cdots = 1$. When n is odd, say $n = 2\lambda + 1$, then $\alpha - \alpha_2 + \alpha_4 - \cdots = 0$, and $\alpha_1 - \alpha_3 + \alpha_5 - \cdots = 1$, when λ is odd; $-\alpha_1 + \alpha_3 - \alpha_5 + \cdots = 1$, when λ is even.

We shall, in particular, consider the case where (15) has the form

$$w^n - v^{2k} \cdot u^{n-2k} = 0, \tag{16}$$

in which n and k must both be either even or odd in order that (16) may reduce to (14) for $u = -1, v = -i$.

After a rather complicated process of elimination the cartesian equation of this special class of curves with the nth roots of unity as foci becomes

$$x^{n-2k} \cdot y^{2k} = (-1)^{n-2k} \cdot \frac{(2k)^{2k}}{n^n \cdot (n-2k)^{2k-n}}, \tag{17}$$

which is an n-ic. For $n = 3, k = 1$, we get the cubic hyperbola $xy^2 = -4/27$. The condition for a proper n-ic in (17) is evidently $n - 2k \geqq 1, n \geqq 3$.

UNIVERSITY OF ILLINOIS.

QUADRATIC SYSTEMS OF CIRCLES IN NON-EUCLIDEAN GEOMETRY.

BY PROFESSOR D. M. Y. SOMMERVILLE.

(Read before the American Mathematical Society, October 26, 1918.)

§ 1. The equation of any circle can be written

$$kS - \alpha^2 = 0, \tag{1}$$

where $S = 0$ is the equation of the absolute, and

$$\alpha \equiv lx + my + nz = 0$$

is the equation of the axis, the center being the absolute polar of this line.

If the circle belongs to a one-parameter system, the coefficients k, l, m, n will be functions of the single parameter λ. The degree of the system is the number of circles which pass through a given point, and its class is the number of circles which touch a given straight line.

If

$$S \equiv a_0x^2 + \cdots + 2f_0yz + \cdots \equiv (S[x)^2 = 0,$$

$$\Sigma \equiv A_0\xi^2 + \cdots + 2F_0\eta\zeta + \cdots \equiv (\Sigma[\xi)^2 = 0$$

represent the point and line equations of the absolute, the line equation of the circle (1) is

$$(2)\quad k\Sigma - (b_0n^2 + c_0m^2 - 2f_0mn)\xi^2 - \cdots + 2(f_0l^2 + a_0mn - h_0nl - g_0lm)\eta\zeta + \cdots = 0,$$

which may also be written

$$\{(\Sigma[l)^2 - k\Delta_0\}\Sigma - \{\Sigma(Al + Hm + Gn)\xi\}^2 = 0.$$

The system of circles will be of the first degree only if l, m, n are constant, and $k = (a\lambda + b)/(c\lambda + d)$; it is then also of the first class. Hence *the only linear system of circles is a concentric system.*

§ 2. If l, m, n are of degree r in λ, while k is of degree $2r$ or $2r + 1$, the system is of degree $2r$ or $2r + 1$ respectively, and its class is also $2r$ or $2r + 1$. Hence *the degree and class are in general the same.*

§ 3. We shall confine our attention now to a quadratic system. The most general form may be written

$$(3)\qquad (p\lambda^2 + 2q\lambda + r)S - (\lambda\alpha + \beta)^2 = 0,$$

where α and β represent distinct fixed lines, and p, q, r are given constants. If the lines α and β coincide, the equation of the system reduces to

$$(p\lambda^2 + 2q\lambda + r)S - (\lambda + k)^2\alpha^2 = 0.$$

For all values of λ this represents a system of circles with axis α, i. e., a concentric system. Though apparently a quadratic system, it is really linear, for two values of λ correspond to each circle.

The lines α and β may be replaced by any two distinct lines through their point of intersection without in any way altering the system; they might, for example, be the tangents to S, or a pair of conjugate lines with regard to S.

The axes of the circles form a pencil of lines $\lambda\alpha + \beta = 0$ through the fixed point $A \equiv (\alpha\beta)$, and the centers are collinear on the line a, the absolute polar of A.

§ 4. *Envelope of the system.* Writing the equation (3) in the form

$$\lambda^2(pS - \alpha^2) + 2\lambda(qS - \alpha\beta) + (rS - \beta^2) = 0,$$

the equation of the envelope is

$$(qS - \alpha\beta)^2 - (pS - \alpha^2)(rS - \beta^2) = 0,$$

i. e.,

$$(q^2 - pr)S^2 + (r\alpha^2 + p\beta^2 - 2q\alpha\beta)S = 0.$$

Hence the envelope consists of the absolute S and a *conic*

$$(4) \qquad K \equiv (pr - q^2)S - (r\alpha^2 - 2q\alpha\beta + p\beta^2) = 0.$$

The equation (3) may be written in the form

$$(5) \quad (p\lambda^2 + 2q\lambda + r)K + \{(q\lambda + r)\alpha - (p\lambda + q)\beta\}^2 = 0,$$

which shows that *every circle of the system has double contact with the conic K, and the chords of contact all pass through A.*

§ 5. *Limiting lines.* In general there are two circles of the system which degenerate to coincident lines, limiting lines. The values of λ are the roots of the equation $p\lambda^2 + 2q\lambda + r = 0$, and the two lines are $r\alpha^2 - 2q\alpha\beta + p\beta^2 = 0$. These form a pair of common chords of S and the conic envelope K, i. e., a pair of focal lines of K. Their point of intersection $A \equiv (\alpha\beta)$ is therefore a center of K, and its absolute polar a is an axis of K. Hence *the line of centers of the circles is one of the axes of the conic envelope, and the point of concurrence of the axes of the circles is the corresponding center of the conic envelope.*

The two limiting lines are a pair of focal lines of K; the centers of these two degenerate circles are a pair of director points of K.

§ 6. *The pairs of corresponding chords of contact of the circles with S and K form an involution whose double lines are the limiting lines.*

To the line $\lambda\alpha + \beta = 0$ corresponds the line

$$(q\lambda + r)\alpha - (p\lambda + q)\beta = 0$$

or, say

$$\mu\alpha + \beta = 0,$$

so that

$$\mu(p\lambda + q) + (q\lambda + r) = 0,$$

i. e.,

$$p\lambda\mu + q(\lambda + \mu) + r = 0,$$

a symmetrical lineo-linear relation. The double elements are then given by

$$p\lambda^2 + 2q\lambda + r = 0, \tag{6}$$

the roots of which correspond to the two limiting lines.

If the roots of the equation (6) are c, d, and the limiting lines are γ, δ, the equation of the system of circles can be written

$$(\lambda - c)(\lambda - d)S - \{\gamma(\lambda - d) - \delta(\lambda - c)\}^2 = 0.$$

Then we find that

$$K \equiv S + 4\gamma\delta,$$

and the equation of the system can be written also

$$(\lambda - c)(\lambda - d)K - \{\gamma(\lambda - d) + \delta(\lambda - c)\}^2 = 0,$$

which verifies that the two chords of contact of any circle of the system with S and K are harmonic conjugates with regard to γ, δ.

From this symmetrical relationship it follows that if one circle has as chords of contact with S and K the lines u and v respectively, there is another circle of the system which has as chords of contact v and u respectively.

§ 7. *Limiting points.* S and K have four common tangents. Any two of these taken together form a degenerate circle having double contact with S and K. But of these pairs just one pair have their centers on the given line of centers of the system of circles. Hence *there are two circles of the system which degenerate to a line pair (point circles) and their centers form a pair of foci of the conic envelope K.* We shall call these two points the *limiting points* of the system.

§ 8. *Associated systems.* Let A, B, C be the three centers, a, b, c the corresponding axes of the conic K, and let f, f'; g, g'; h, h' be the focal lines through A, B, C respectively; F, F'; G, G'; H, H' the foci on a, b, c respectively. Then

there are three systems of circles, one with limiting points F, F', and limiting lines f, f', the second with limiting points G, G', etc.

§ 9. *Horocycles.* In general the system contains two horocycles, corresponding to the two values of λ which make the line $\lambda\alpha + \beta = 0$ a tangent to S. These values of λ are the roots of the quadratic

$$\lambda^2(\Sigma[l_1)^2 + 2\lambda(\Sigma[l_1][l_2) + (\Sigma[l_2)^2 = 0.$$

§ 10. *Degenerate systems.* The discriminant of the conic envelope

$$K \equiv (pr - q^2)S - (r\alpha^2 - 2q\alpha\beta + p\beta^2) = 0,$$

where

$$S \equiv (a_0b_0c_0f_0g_0h_0[xyz)^2,$$

and

$$\alpha \equiv l_1x + m_1y + n_1z, \qquad \beta \equiv l_2x + m_2y + n_2z,$$

is

$$(7)\quad (pr - q^2)^2[\Delta_0(pr - q^2) - p(\Sigma[l_2)^2 + 2q(\Sigma[l_1)(l_2) - r(\Sigma[l_1)^2 + (S[m_1n_2 - m_2n_1)^2] = 0,$$

Δ_0 being the discriminant of S.

If $pr - q^2 = 0$, the equation of the conic envelope reduces to

$$r\alpha^2 - 2q\alpha\beta + p\beta^2 = 0,$$

which now represents two coincident lines. As an envelope, the conic degenerates to two points. Put $p = k^2$, $q = k$, $r = 1$; then the equation of the system of circles (3) becomes

$$(k\lambda + 1)^2S - (\lambda\alpha + \beta)^2 = 0,$$

i. e.,

$$\lambda^2(k^2S - \alpha^2) + 2\lambda(kS - \alpha\beta) + (S - \beta^2) = 0.$$

Now

$$k^2S - \alpha^2 \equiv k^2(S - \beta^2) + (k^2\beta^2 - \alpha^2),$$

and

$$kS - \alpha\beta \equiv k(S - \beta^2) + \beta(k\beta - \alpha).$$

Hence the three conics $k^2S - \alpha^2 = 0$, $kS - \alpha\beta = 0$, $S - \beta^2 = 0$ have two points in common, their common chord being $\alpha - k\beta = 0$. All the circles of the system therefore pass through these two fixed points, and this point pair is the

degenerate conic envelope K. We have then a *coaxal system* of circles.

When the other factor of the discriminant (7) vanishes, the conic envelope reduces to two distinct lines, and we have a *homocentric system* of circles with two common tangents. Multiplying the factor by Δ_0, the condition can be expressed in the form

$$[\Delta_0 p - (\Sigma l_1)^2][\Delta_0 r - (\Sigma l_2)^2] - [\Delta_0 q - (\Sigma l_1 l_2)]^2 = 0.$$

§ 11. The system will be specialized also if the conic envelope is a circle. If K is a circle the common chords $r\alpha^2 - 2q\alpha\beta + p\beta^2 = 0$ must either coincide or be tangents to the absolute. In the former case, however, K degenerates to these two coincident straight lines, or, as an envelope, to a point pair.

The equation of the tangents from the point $(\alpha\beta)$ to the absolute is

$$\alpha^2(\Sigma l_2)^2 - 2\alpha\beta(\Sigma l_1 l_2) + \beta^2(\Sigma l_1)^2 = 0.$$

Hence the conditions that K should be a circle are

$$p = k(\Sigma l_1)^2, \qquad q = k(\Sigma l_1 l_2), \qquad r = k(\Sigma l_2)^2.$$

Then

$$pr - q^2 = k^2\Delta_0(S m_1 n_2 - m_2 n_1)^2,$$

and the discriminant of K, (7), becomes

$$k^4\Delta_0^2[(S m_1 n_2 - m_2 n_1)^2]^3\ (k\Delta_0 - 1)^2.$$

In this case therefore K does not in general degenerate further unless the point $(\alpha\beta)$ lies on the absolute. Hence *the conic envelope may be a circle, but cannot be a horocycle.*

When K is a circle its axis is the line of centers of the circles of the system. The common chords with S and K, being harmonic conjugates with regard to the two common tangents of S and K, are at right angles. The chords of contact with K all pass through $(\alpha\beta)$ and are therefore perpendicular to the axis of K. Hence all the circles of the system have equal radii. The system therefore consists of a system of equal circles with collinear centers. The limiting points reduce to the two absolute points of K, and the limiting lines to the absolute lines of K.

§12. Taking two conjugate lines through $(\alpha\beta)$ and the absolute polar of $(\alpha\beta)$ as the triangle of reference, the equation of the general quadratic system of circles can be written

$$(p\lambda^2 + 2q\lambda + r)(x^2 + y^2 + z^2) - (\lambda x + y)^2 = 0.$$

Let the radius be ρ, then

$$(8) \quad c^2 = \cos^2 \rho = \frac{(\lambda x + y)^2}{(x^2 + y^2 + z^2)(\lambda^2 + 1)} = \frac{p\lambda^2 + 2q\lambda + r}{\lambda^2 + 1}.$$

Hence *there are in general two circles of the system with a given radius.*

Let the parameters of the two circles be λ_1 and λ_2, which are the roots of the equation

$$(9) \qquad (p - c^2)\lambda^2 + 2q\lambda + (r - c^2) = 0.$$

Then

$$\lambda_1 + \lambda_2 = -2q/(p - c^2),$$
$$\lambda_1\lambda_2 = (r - c^2)/(p - c^2).$$

Eliminating c^2, we get

$$q\lambda_1\lambda_2 - \tfrac{1}{2}(p - r)(\lambda_1 + \lambda_2) - q = 0.$$

Hence *the parameters of pairs of equal circles are connected by an involutorial homographic relation.*

There are *two coincident pairs of equal circles,* whose parameters are the roots of the equation

$$q\lambda^2 - (p - r)\lambda - q = 0,$$

and it is easily verified that these are the circles of maximum or minimum radius. The radii of these two circles are found by expressing the condition for equal roots in the equation for λ, (9); we get then the equation

$$c^4 - (p + r)c^2 + (pr - q^2) = 0.$$

if these two values of the radius are equal, all the radii will be equal. The conditions for this are $q = 0$ and $p = r$. Equation (8) cannot then be satisfied except for a particular value of c.

If there are three circles of the system with equal radii, all the circles of the system must have equal radii.

§ 13. *Two-parameter system of circles.* There is no linear two-parameter system of circles. The general quadratic system can be written in terms of the homogeneous parameters λ, μ, ν

$$(10)\quad (a\lambda^2 + b\mu^2 + c\nu^2 + 2f\mu\nu + 2g\nu\lambda + 2h\lambda\mu)S - (\lambda\alpha + \mu\beta + \nu\gamma)^2 = 0,$$

where α, β, γ represent three straight lines.

If the lines α, β, γ are concurrent, so that $\gamma \equiv l\alpha + m\beta$, then, putting $\lambda + l\nu = \lambda'$, $\mu + m\nu = \mu'$, the equation can be reduced to the form

$$(a'\lambda'^2 + \cdots + 2f'\mu'\nu' + \cdots)S - (\lambda'\alpha + \mu'\beta)^2 = 0.$$

This represents a system of circles in which the only restriction is that the center lies on a fixed straight line, the polar of $(\alpha\beta)$.

If α, β, γ are coincident the circles are concentric with center the pole of α.

Passing to the general case in which α, β, γ form a triangle, we can transform the equation by a linear transformation so that the triangle α, β, γ is transformed into a self-conjugate triangle of the absolute. Taking this as triangle of reference the equation can be written in the form

$$(11)\qquad P_1S - (\lambda x + \mu y + \nu z)^2 = 0,$$

where

$$S \equiv x^2 + y^2 + z^2,$$

and

$$P_1 \equiv a\lambda^2 + b\mu^2 + c\nu^2 + 2f\mu\nu + 2g\nu\lambda + 2h\lambda\mu.$$

The center of the circle is therefore (λ, μ, ν). When $\lambda : \mu : \nu$ are given, the circle is fixed, hence, *with a given point as center there is one and only one circle of the system.*

The line equation of the circle (11) is found to be

$$(\lambda^2 + \mu^2 + \nu^2 - P_1)(\xi^2 + \eta^2 + \zeta^2) - (\lambda\xi + \mu\eta + \nu\zeta)^2 = 0,$$

or

$$(12)\qquad P_0\Sigma - (\lambda\xi + \mu\eta + \nu\zeta)^2 = 0.$$

§ 14. *Locus of limiting points and envelope of limiting lines.* The discriminant of (11) is

$$\begin{vmatrix} P_1 - \lambda^2 & -\mu\lambda & -\nu\lambda \\ -\lambda\mu & P_1 - \mu^2 & -\nu\mu \\ -\lambda\nu & -\mu\nu & P_1 - \nu^2 \end{vmatrix} = 0$$

i. e.,

$$P_1^2(P_1 - \lambda^2 - \mu^2 - \nu^2) = 0 \quad \text{or} \quad P_1^2 P_0 = 0.$$

If $P_1 = 0$ the circle (11) degenerates to two coincident lines $\lambda x + \mu y + \nu z = 0$. The line equation of the envelope of these limiting lines is therefore

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta = 0.$$

The point equation of the envelope of limiting lines is therefore

(13) $S_1' \equiv Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0.$

The locus of the absolute poles of these lines, or centers of the circles which degenerate to coincident lines, is the reciprocal of S_1' with regard to S, and its point equation is

(14) $S_1 \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$

The circle (12) degenerates to two distinct straight lines, or a point circle, if $P_0 = 0$. The center being (λ, μ, ν), the equation of the locus of point circles is therefore

(15) $$S_0 \equiv S - S_1 = 0.$$

The envelope of the axes of the point circles is the reciprocal of (15) with regard to S, and its point equation is thus found to be

(16) $$S_0' \equiv S_1 + S_1' + (1 - \Sigma a)S = 0.$$

§ 15. *Horocycles.* The circle will be a horocycle if the line $\lambda x + \mu y + \nu z = 0$ is a tangent to S, i. e., if

$$\lambda^2 + \mu^2 + \nu^2 = 0.$$

Hence through any point there pass four horocycles of the system.

§ 16. Consider the circles of the system whose centers lie on a given line $lx + my + nz = 0$. We shall call this a *row of circles*.

We have

$$l\lambda = m\mu + n\nu = 0.$$

Eliminating ν, we get

$$[n^2(a\lambda^2 + 2h\lambda\mu + b\mu^2) - 2n(g\lambda + f\mu)(l\lambda + m\mu) + c(l\lambda + m\mu)^2]S - [n(\lambda x + \mu y) - (l\lambda + m\mu)z]^2 = 0.$$

This is a one-parameter system of circles

$$(17)\quad [(an^2 - 2gnl + cl^2)\lambda^2 + 2(hn^2 - gmn - fnl + clm)\lambda\mu + (bn^2 - 2fmn + cm^2)\mu^2]S - [(nx - lz)\lambda + (ny - mz)\mu]^2 = 0,$$

having as envelope the conic

$$(18)\quad T \equiv (Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm)S - \Sigma x^2(cm^2 - 2fmn + bn^2) + 2\Sigma yz(fl^2 + amn - hnl - glm) = 0.$$

This is a two-parameter quadratic system of conics. Two of the foci of T are limiting points of the system of circles and therefore lie on the conic S_0, and the corresponding directrices touch the conic S_0'; the two corresponding focal lines are limiting lines of the system and therefore touch the conic S_1', and the corresponding director points lie on the conic S_1.

§ 17. The two lines

$$\Sigma x^2(cm^2 - 2fmn + bn^2) - 2\Sigma yz(fl^2 + qmn - hnl - glm) = 0,$$

which pass through the point of intersection of $nx - lz = 0$ and $ny - mz = 0$, i. e., the point (l, m, n), are in fact the tangents from this point to the conic S_1', and the equation of the conic envelope can be written

$$\Delta T \equiv (Al^2 + \cdots + 2Fmn + \cdots)(\Delta S - S_1') - [x(Al + Hm + Gn) + \cdots]^2 = 0.$$

Hence *the conic envelopes T all have double contact with the fixed conic $\Delta S - S_1' = 0$.*

If the conic S' coincides with the absolute S, the conics T are all circles. In this case all the circles of the system have equal radii.

We shall consider in the next section what rows of circles with equal radii exist in the general system.

§ 18. Let ρ be the radius of the circle

$$P_1(x^2+y^2+z^2)-(\lambda x+\mu y+\nu z)^2=0.$$

Then

$$\cos^2\rho=\frac{(\lambda x+\mu y+\nu z)^2}{(x^2+y^2+z^2)(\lambda^2+\mu^2+\nu^2)}=\frac{P_1}{\lambda^2+\mu^2+\nu^2}.$$

If the center (λ, μ, ν) is given, the radius is in general determined, so that with a given center there is in general just one circle of the system. ρ will, however, be indeterminate if both

$$P_1=0 \quad \text{and} \quad \lambda^2+\mu^2+\nu^2=0.$$

Hence if the center is at one of the four absolute points on the conic S_1 or S_0 the radius is indeterminate.

Consider the locus of centers of circles with a given radius. The equation of the locus is

$$(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)-(x^2+y^2+z^2)\cos^2\rho=0.$$

Hence the locus is a conic homothetic with S_1 and S_0.

Putting $\cos^2\rho=r$, the values of r for which this conic breaks up into a pair of straight lines are the roots of the equation

$$\begin{vmatrix} a-r & h & g \\ h & b-r & f \\ g & f & c-r \end{vmatrix}=0$$

This equation has three real roots, hence *there are three pairs of rows of circles each with equal radii.* We shall call the lines of centers of these circles the *equiradial axes.*

The three pairs of equiradial axes are common chords of S and S_1, i. e., the pairs of focal lines of the conic S_1 (or S_0). The axes of these rows of circles are concurrent in three pairs of points which are the director points of S_1 (or S_0), or the foci of S_1' (or S_0').

These results are easily proved also directly by taking as the triangle of reference the common self-conjugate triangle of S and S_1, so that the equation of the system becomes

$$(a\lambda^2+b\mu^2+c\nu^2)(x^2+y^2+z^2)-(\lambda x+\mu y+\nu z)^2=0.$$

The three pairs of equiradial axes are then given by

$$(a-b)y^2-(c-a)z^2=0,$$
$$(b-c)z^2-(a-b)x^2=0,$$
$$(c-a)x^2-(b-c)y^2=0,$$

the radii being respectively $\cos^{-1}\nu a$, $\cos^{-1}\nu b$, $\cos^{-1}\nu c$.

If $b=c$, the conics S_1 and S_0 have double contact with S and are concentric circles. In this case the last two pairs of axes coincide with $x=0$, which is the axis of S_1 or S_0, and the first pair reduces to $y^2+z^2=0$, which consists of the absolute tangents of S_1 or S_0.

§ 19. The discriminant of the conic envelope T is

$$\Sigma_1^2\Sigma_0=0,$$

where

$$\Sigma_1\equiv\Sigma Al^2+2\Sigma Fmn,$$
$$\Sigma_0\equiv(\Sigma Al^2+2\Sigma Fmn)+(\Sigma al^2+2\Sigma fmn)-\Sigma l^2(\Sigma a-1).$$

When $\Sigma_1=0$, T degenerates to a point pair. Since $\Sigma_1=0$ is the line equation of the conic S_1, we see that *the circles whose centers all lie on a tangent to S_1 form a coaxial system*, the common radical axis being the absolute polar of the point of contact of this tangent.

When $\Sigma_0=0$, T degenerates to a line pair. Since $\Sigma_0=0$ is the line equation of the conic S_0, we see that *the circles whose centers all lie on a tangent to S_0 form a homocentric system*, the common homothetic center being the point of contact of this tangent.

§ 20. T will degenerate to a point pair for every row of circles if Σ_1 vanishes identically, i. e., if P_1 is a perfect square. Then all the limiting lines pass through one point, and the locus of point circles S_0 becomes a circle with this point as center. In this case we can prove that *every circle of the system cuts the circle S_0 orthogonally.*

Let $a=p^2$, $b=q^2$, $c=r^2$, $f=qr$, $g=rp$, $h=pq$, then

$$-T\equiv\Sigma x^2(rm-qn)^2+2\Sigma yz(pn-rl)(ql-pm)$$
$$\equiv[\Sigma(rm-qn)x]^2.$$

Hence the point pair lies on the line $\Sigma(rm-qn)x=0$ which

passes through the fixed point (p, q, r), the center of the circle S_0.

The polar of (p, q, r) with regard to the circle

$$(\Sigma p\lambda)^2 \cdot \Sigma x^2 - (\Sigma \lambda x)^2 = 0$$

is

$$\Sigma p\lambda \cdot \Sigma px - \Sigma \lambda x = 0.$$

Now forming the equation of a pair of common chords of this circle and the circle

$$S_0 \equiv (\Sigma px)^2 - \Sigma x^2,$$

we have

$$(\Sigma p\lambda)^2 \cdot \Sigma x^2 - (\Sigma \lambda x)^2 + [(\Sigma px)^2 - \Sigma x^2](\Sigma p\lambda)^2 = (\Sigma px)^2(\Sigma p\lambda)^2 - (\Sigma \lambda x)^2.$$

Hence one of the common chords is $\Sigma px \cdot \Sigma p\lambda - \Sigma \lambda x = 0$, which is the polar of (p, q, r). Hence the circles cut orthogonally.

§ 21. T will degenerate to a line pair for every row of circles of the system if Σ_0 vanishes identically. This is the condition that P_0 should be a perfect square.

Now every point on a circle is tangentially distant a quadrant from its axis, and if two circles are such that the length of one of their common tangents is a quadrant, say the circles are *quadrantal*, each of the points of contact of this tangent must lie on the axis of the other circle. The same will be true for the other common tangent belonging to the same pair. If these two tangents intersect in 0, the polar of 0 with regard to each circle is the axis of the other circle.

Since the line equation of the system of circles is

$$P_0\Sigma - (\lambda\xi + \mu\eta + \nu\zeta)^2 = 0,$$

the analytical work in deducing the conclusions from the conditions that P_0 should be a perfect square is exactly the same as that of the last section. Hence if this condition is satisfied, *the locus of point circles consists of two coincident straight lines, the envelop of limiting lines is a circle, of which the locus of point circles is the axis, and every circle of the system is quadrantal with respect to this circle.*

VICTORIA UNIVERSITY COLLEGE,
WELLINGTON, N. Z.,
August, 1918.

CONTINUOUS SETS THAT HAVE NO CONTINUOUS SETS OF CONDENSATION.

BY PROFESSOR R. L. MOORE.

(Read before the American Mathematical Society, October 27, 1917.)

JANISZEWSKI has shown* that if A and B are two distinct points then every bounded set of points that is *irreducibly* continuous† from A to B, and has no continuous set of condensation,† is a simple continuous arc from A to B. In the present paper I will establish the following result.

THEOREM. *Every bounded continuous set of points that has no continuous set of condensation is a continuous curve.*‡

Proof. Suppose M is a bounded continuous set of points that has no continuous set of condensation. It has been shown by Hahn§ that every bounded continuous set of points that is connected "im kleinen" is a continuous curve. I shall proceed to show that the point set M is connected "im kleinen." Suppose that it is not. Then there is a point P belonging to M and a circle K with center at P such that within every circle whose center is P there exists a point which does not lie together with P in any connected subset of M that lies

* S. Janiszewski, "Sur les continus irréductibles entre deux points," *Journal de l'Ecole Polytechnique*, 2e série, vol. 16 (1911–12), pp. 79–170.

† A set of points is said to be *connected* if, however it be divided into two mutually exclusive subsets, one of these subsets contains a limit point of the other one. A set of points is said to be *continuous* if it is closed and connected and contains more than one point. A continuous set of points containing the two distinct points A and B is said to be *irreducibly continuous* from A to B if it contains no other continuous set that contains both A and B. The continuous set N is said to be a *continuous set of condensation* of the continuous set M if N is a subset of M and every point of N is a limit point of $M-N$.

‡ A *continuous curve* is the set of all points $\{(x, y)\}$ satisfying the equations $x = f_1(t)$, $y = f_2(t)$ $(0 \leqq t \leqq 1)$, where $f_1(t)$ and $f_2(t)$ are continuous functions of t. In case there do not exist, in the interval $(0 \leqq t \leqq 1)$, two distinct numbers t_1 and t_2 such that $f_1(t_1) = f_1(t_2)$ and $f_2(t_1) = f_2(t_2)$, then this curve is a *simple* continuous arc.

§ Hans Hahn, "Ueber die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 23 (1914), pp. 318–322. A set of points M is said to be *connected "im kleinen"* (cf. Hahn, loc. cit.) if for each point P of M and each circle K with center at P there exists, within K, another circle K', with center at P, such that if X is a point of M within K' then X and P lie together in some connected subset of M that lies entirely within K.

entirely within K. Let $K_1, K_2, K_3, \cdots$ denote an infinite sequence of circles with center at P such that the radius of K_n is $r/2n$, where r is the radius of K. For every n there exists, within K_n, a point X_n such that X_n and P do not lie together in any connected subset of M that lies entirely within K. Let $\overline{K}$ denote a definite circle, with center at P, that lies within K and encloses K_1. By a theorem due to Janiszewski,* M contains at least one point set that is irreducibly continuous from X_n to P. Let t_n denote one such set. The set t_n is a subset of M and M has no continuous set of condensation. It follows that t_n has no continuous set of condensation. Hence, by the first of the above mentioned theorems of Janiszewski, t_n is a simple continuous arc† from X_n to P. This arc can not lie entirely within K. Let Y_n denote the first point, in the order from X_n to P, that it has in common with $\overline{K}$. The interval X_nY_n of the arc t_n lies, except for the point Y_n, entirely within $\overline{K}$. There exists n_1 such that if $n \geqq n_1$ then X_nY_n has no point in common with X_1Y_1. For otherwise there would exist an infinite subsequence $\overline{X_1Y_1}$, $\overline{X_2Y_2}$, $\overline{X_3Y_3}$, $\cdots$ of the sequence $X_1Y_1, X_2Y_2, X_3Y_3, \cdots$ such that, for every n, $\overline{X_nY_n}$ has a point in common with X_1Y_1. The point set composed of the point P together with the sum of the arcs X_1Y_1, $\overline{X_1Y_1}$, $\overline{X_2Y_2}$, $\overline{X_3Y_3}$, $\cdots$ would then be a connected point set lying within K and containing both P and X_1. Thus a contradiction would be obtained. Similarly there exists n_2, greater than n_1, such that if $n \geqq n_2$ then X_nY_n has no point in common with $X_{n_1}Y_{n_1}$. There exists n_3 greater than n_2 such that if $n > n_3$ then X_nY_n has no point in common

* S. Janiszewski, "Sur la géometrie de lignes cantoriennes," *Comptes Rendus*, vol. 151 (1910), pp. 198–201.

† It may be of interest to note that in order that a bounded continuous set of points M should be a continuous curve it is not sufficient that every two points of M should be the extremities of a simple continuous arc lying wholly in M. To see this consider the following example.

Example. Let AB denote the interval from (0, 0) to (0, 1), in a rectangular system of coordinates, and let $B_1, B_2, B_3, \cdots$ denote the points $(1, 1), (1, \frac{1}{2}), (1, \frac{1}{3}), \cdots$ respectively. The point set composed of the intervals $AB, AB_1, AB_2, AB_3, \cdots$ is a bounded, continuous set of points M every two points of which can be joined by an arc that lies in M. But M is not connected "im kleinen" and is therefore not a continuous curve.

However in my paper "A theorem concerning continuous curves," this BULLETIN, vol. 23 (1917), pp. 233–236, I proved the truth of the converse proposition that every two points of a continuous curve are the extremities of at least one simple continuous arc lying in the point set constituted by that curve.

with $X_{n_2}Y_{n_2}$. If this process is continued there will be obtained an infinite sequence of arcs $X_{n_1}Y_{n_1}$ $X_{n_2}Y_{n_2}$, $X_{n_3}Y_{n_3}$, $\cdots$ no two of which have any point in common. For each i, the arc $X_{n_i}Y_{n_i}$ contains, as a subset, an arc $W_{n_i}Y_{n_i}$ which lies between the circles $\overline{K}$ and K_1, except for the points W_{n_i} and Y_{n_i} which lie on K_1 and $\overline{K}$ respectively. There exist 1) on $\overline{K}$ an infinite sequence of distinct points Y', Y_1', Y_2', Y_3', $\cdots$, 2) on K_1 an infinite sequence of distinct points W', W_1', W_2', W_3', $\cdots$, 3) an infinite sequence of distinct arcs $W_1'Y_1'$, $W_2'Y_2'$, $W_3'Y_3'$, $\cdots$ all belonging to the set $W_{n_1}Y_{n_1}$, $W_{n_2}Y_{n_2}$, $W_{n_3}Y_{n_3}$, $\cdots$, such that Y' is the sequential limit point of the sequence Y_1', Y_2', Y_3', $\cdots$ and W' is the sequential limit point of the sequence W_1', W_2', W_3', $\cdots$. No two of the arcs $W_1'Y_1'$, $W_2'Y_2'$, $W_3'Y_3'$, $\cdots$ have a point in common. It easily follows that there exists a closed connected point set N, containing Y' and W', such that every point of N is a limit point of the point set constituted by the sum of the arcs $W_1'Y_1'$, $W_2'Y'_2$, $W_3'Y_3'$, $\cdots$. The point set N is a continuous set of condensation of the set M.

Thus the supposition that M is not connected "im kleinen" leads to a contradiction. It follows that M is a continuous curve.

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA.

DERIVATIVELESS CONTINUOUS FUNCTIONS.

BY PROFESSOR M. B. PORTER.

(Read before the American Mathematical Society October 26, 1918.)

THERE is no more interesting illustration of the refinement of geometric intuition through the influence of the arithmetization of mathematics than that presented by the history of functions of this type. No less a mathematician than Ampère, not to mention Duhamel and Bertrand, thought he had actually proved that continuous functions had derivatives for all save a finite number of arguments. Darboux in his paper on "Discontinuous functions" published in the *Annals* of the Ecole Normale for 1875, though dated January 20, 1874, in

connection with his example of a function of this class mentions only one auditor, M. Bienaimé, who said that he was unconvinced by Ampère's proof. Darboux makes no mention of Weierstrass's work published in 1874 in *Crelle* on the function $\Sigma a^n \cos b^n x$ and was doubtless not aware of it. Du Bois-Reymond was so awestruck by Weierstrass's curve as to provoke some rather jocular remarks by Wiener in the introduction to his paper on Weierstrass's curve in the 90th volume of *Crelle's Journal*.

It does not seem to have been noticed by writers who have considered these functions, especially of Weierstrass's type, that a considerable simplification of treatment was possible by a more obvious choice of the δx used in the incremental ratio and that other advantages might result from such an innovation.

We begin by treating in detail a Weierstrass function

$$W(x) = \sum_0^\infty a^n \sin b^n \pi x \qquad |a| < 1,\ b \text{ integral.}$$

Setting

$$\delta x = 2k/b^{N+1}, \quad k \text{ integral},$$

we get by applying the mean value theorem to the first N terms and a trigonometric identity to the last term,

$$(1) \quad \frac{\Delta W(x)}{\delta x} = \pi \sum_0^{N-1} (ab)^n \cos b^n \pi (x + \theta \delta x) + \pi (ab)^N \sin \frac{k\pi}{\pi k/b} \cos\left(b^n x + \frac{k}{b}\right)\pi.$$

Evidently the absolute value of the first N terms is less than

$$\pi \sum_0^{n-1} |ab|^n < \frac{\pi |ab|^n}{|ab| - 1}$$

if $|ab| > 1$. As to the last term in (1), if we suppose that $k \equiv \pm \frac{3}{4} b$, which implies that 4 is a factor of b, it is, in absolute value,

$$(2) \qquad \geqq \tfrac{1}{4}\pi |ab|^N \frac{1}{\frac{3}{4}\pi\sqrt{2}} \left|\cos\left(b^n x + \frac{k}{b}\right)\pi\right|$$

In this expression we can find two values of k, k_1 and k_2, such that

$$1°. \quad 1 \geqq \cos\left(b^n x + \frac{k_1}{b}\right)\pi \geqq \frac{1}{\sqrt{2}}$$

$$2°. \quad -\frac{1}{\sqrt{2}} \geqq \cos\left(b^n x + \frac{k_2}{b}\right)\pi \geqq -1$$

where k_1 and k_2 have *opposite** signs.

Thus for these values of k

$$\left|\pi(ab)^N \frac{1}{\frac{3}{4}\pi\sqrt{2}} \cos\left(b^N x + \frac{k}{b}\right)\pi\right| \geqq \frac{\pi|ab|^N}{\frac{3}{2}\pi}.$$

The last term of (1) will thus dominate in sign and magnitude the first N terms if

$$|ab| > 1 + \tfrac{3}{2}\pi.$$

Hence the right and left incremental ratio which we are considering will become infinite with N but will always have opposite signs.

This proves that $W(x)$ has a derivative for no value of x.

Making use of the remark at the end of the footnote, we can prove that, except possibly for a set of x-points of Borel measure zero (null set), $W(x)$ has neither a backward nor a forward tangent.

To do this suppose x written in system radix b, i. e.,

$$x = b_0 + \frac{b_1}{b} + \frac{b_2}{b^2} + \cdots,$$

where b_1, b_2, etc., $< b$. All those x's in which any one of the digits $0, 1, \cdots, b-1$ fails to occur infinitely often form a null set. Hence, except for the points of the null set $[x]$

* This can be seen at once from a diagram. Divide the angle 2π into 8 equal parts. If for example $b^N x\pi$ lies in the first octant, one and possibly more values of k can be found between 1 and $\frac{1}{4}b$ so that 1° holds; while one or possibly more values lie between 0 and $-b/4$ for which 2° holds. Similarly for the other octants.

Moreover it is clear that if $b^N x\pi$ has its terminal line between $I\pi$ and $(I + 1/b)\pi$, I integral or zero, a positive k_1 can be found for which 1° holds and another positive k_2 for which 2° holds. If $b^N \pi x$ lies between $(I + (b-1)/b)\pi$ and $(I+1)\pi$, two negative k's can be found for which 1° and 2° hold respectively.

for which $b_i = 0$ and $b_i = b - 1$ fail to occur infinitely often, the right and left incremental ratios have each for their upper and lower limits $+\infty$ and $-\infty$ respectively. The set $[x]$ is made up of a countable number of perfect null sets and is everywhere dense on the line; it is consequently not closed.

At the points of $[x]$ $W(x)$ might possibly have vertical cusps but not elsewhere.

It will be noted that the signs of the individual terms of our series are of no significance and hence can be arbitrarily changed to $+$ or $-$. This conclusion does not seem to follow from Weierstrass's proof.

Weierstrass (also Dini and others) unnecessarily require that b be an odd integer; our restriction that b be a multiple of 4 can evidently be removed if $|ab| > 9$, and the proof just given will remain valid.

The formation of an extensive class of such functions

$$W(x) = \sum_0^\infty u_n(x) \sin i_n x\pi \quad \text{or} \quad \sum_0^\infty u_n(x) \cos i_n x\pi,$$

i_n integral, is easy. We require:

1°. Uniform convergence when $l \leqq x \leqq L$.

2°. i_n must divide i_{n+1} and for an unlimited number of n's their ratio must either be divisible by 4 or increase indefinitely with n.

3°. $\sum_0^\infty u_n'(x)$ must be uniformly convergent by Weierstrass's G test, $l \leqq x \leqq L$.

4°. $\frac{3}{2}\pi \sum_0^{N-1} |i_n u_n(x)| < |i_N u_N(x)|$ in the interval of convergence of $W(x)$.

These conditions are merely sufficient and 2° and 4° admit of certain obvious modifications. Functions which fulfill these conditions will have no derivatives in the interval (lL).

The following specimens will suffice:

1. $\sum_0^\infty \frac{a^n}{n!} \sin (n!\pi x) \cos \qquad |a| > 1 + \frac{3}{2}\pi.$

2. $\sum_1^\infty \frac{a^n}{1\cdot 3\cdot 5\cdots(2n-1)} \cos(1\cdot 3\cdot 5\cdots 2n-1)\pi x$

$|a| > 1 + \frac{3}{2}\pi$ (Dini).

3. $\sum_1^\infty \frac{a^n}{1\cdot 5\cdot 9\cdots(4n+1)} \sin(1\cdot 5\cdot 9\cdots(4n+1))\pi x$

$a > 1 + \frac{3}{2}\pi$ (Dini).

4. If $\sum_1^\infty \frac{a_i}{10^i}$ denote any non-terminating decimal,

$$\sum_0^\infty \frac{a_i}{10^i} \sin\cos(10^{3i}x\pi).$$

5. $\sum_0^\infty \frac{1}{a^n} \sin\cos(n!a^n\pi x)$, where $|a|$ is an integer > 1.

6. $x\sum_0^\infty a^n \sin b^n x\pi$, $|a|<1$, $|ab|>1+\frac{3}{2}\pi$, has a derivative for $x = 0$ but for no other value of x.

7. $\sum_0^\infty \frac{x^n}{n!} \sin\cos(n!\pi x)$ has derivatives *between* -1 and $+1$ and no derivatives if $|x| > 1 + \frac{3}{2}\pi$.

Lerch gives a theorem* which shows that this last function has no derivatives for any *rational* points for which $|x| \geqq 1$. It is easy to show that it can have a finite derivative for no point $|x| > 1 + \frac{\pi}{2}$.

A HALF CENTURY OF FRENCH MATHEMATICS.

Les Sciences Mathématiques en France depuis un Demi-Siècle. Par EMILE PICARD. Paris, Gauthier-Villars, 1917. 24 pp.

IN the first decades of the last century the home of the scientific spirit was in France. Paris was the capital of the Republic of exact truth. Interest in scientific discovery and creation was widespread among her people. The spirit of literature flourished alongside the spirit of exact researches

* Lerch, *Crelle's Journal*, vol. 103, p. 130 ("Ueber die Nichtdifferentiierbarkeit gewisser Funktionen").

and both found place in the same creative intellect. Out of this union of elements, too much separated in other countries, there grew up a tradition of literary excellence in scientific exposition which abides to the present and contributes in no small way to the comfort and delight which every one must feel in reading a French scientific book or memoir.

From the extreme precision and abstract character of mathematics it probably does not lend itself as readily to literary form as most other sciences; but, even here, the French have found a method of presentation which renders delightful both the books and memoirs of her masters. On account of their preëminence in this respect it is probably this which first comes to mind when one thinks of the last half century of French mathematics.

Yet the actual and detailed contributions of French investigators to the body of recent discoveries in mathematics looms large in quantity and bears the distinguishing mark of the highest quality. Indeed, leading mastery in the art of presentation can grow out of nothing other than a penetrating insight into the secrets of a subject.

Judged in the light of the excellence of French exposition that which is current in America is seen to fall well below the standard to which we should seek to attain. Among us there is a dearth of expository treatments, both of the sort by which one may get his introduction to a chapter of modern mathematics and of that by which he may obtain a well-rounded initial view of a comprehensive subject. Our memoirs are often printed in such condensed form that even practised mathematicians must waste in reading them time which might have been saved by the use of larger space in setting forth a more illuminating exposition. There is also a tendency towards the publication of matter which is not sufficiently well digested in thought. Results at the half-way stage towards the goal of an investigation are harder to understand than those of the ultimate reach which come of a more penetrating analysis. Too many memoirs appear profound because the writers have not discovered the simpler and more fundamental truths. The French are at present the leading masters in avoiding this defect and attaining the opposite virtue.

The production of work of great value requires extended continuity of spirit and effort. In recent French mathematics

this truth is illustrated by the intimate relation of later discoveries to the earlier work of Fourier, Cauchy, and Galois. A profound study of nature was the source of the fruitful mathematical discoveries of Fourier, who found in physics the first origins of his great analytical theories and who developed analysis primarily for its usefulness to him as a physicist. His work, admirable for its clarity, contained the germ of important methods since employed in the fuller development of the theory of differential equations and of the expansion of arbitrary functions. At the opposite extreme was Galois with his investigation of the theory of algebraic equations and their groups. His researches were far removed from practical experience and even from contact with any theory of nature. He was engaged in problems belonging to the most abstract of pure mathematical disciplines. Between these two and reaching to both extremes was found the prodigious activity of Cauchy, whose researches extended to every domain of pure and applied mathematics. His greatest creation was that of the theory of functions of a complex variable.

During the past hundred years—and no less in its latter half than in its former—the theory of functions of a complex variable has been in great honor among the French. To an extent unusual in our generation we have here a vast subject dominated by the spirit of a single people. It is significant of the character of French thought that this theory of functions of a complex variable is perhaps the most elegant theory of like extent in the whole domain of mathematics.

The pamphlet under review is reprinted from a work published in 1916 with the title Un Demi-Siècle de Civilisation française (1870–1915) and thus covering the period between the present and the last great war fought by France. After an introduction of three pages, Picard gives here a brief analysis of the French mathematics of the period, presenting his matter under the following headings: analytic functions (pages 4–9); differential equations (pages 9–14); theory of numbers, algebra and geometry (pages 14–19); theory of functions of real variables and theory of assemblages (pages 19–23); some final remarks (pages 23–24).

During the last fifty years an important part of French mathematical effort has been devoted to the theory of functions both in its general aspects and as regards certain special classes of functions. Among those who have been interested

in the general theory we may cite the names of Laguerre, Poincaré, Picard, Appell, Goursat, Painlevé, Hadamard, Borel. The demonstration of the Cauchy theorem concerning the zero value of an integral along a contour had formerly assumed the continuity of the derivative; Goursat has shown that this hypothesis is unnecessary. Picard has established the fact that in the neighborhood of an isolated essential singularity a function takes infinitely often every given value with the possible exception of at most *two* values; many authors have drawn various consequences from this remarkable theorem. We may also call attention to the developments, in series of polynomials, of Appell and Painlevé, to the functions with lacunary spaces of Poincaré and Goursat, and to the more recent developments of Montel. Concerning the fundamental character of a power series on the circle of convergence are to be found Hadamard's classic and far-reaching investigations. Laguerre's study of entire functions, Borel's treatment of summable series, and Poincaré's work on multiform functions are other matters to which attention should be directed.

In our period the most important work dealing with special classes of functions has been that of Poincaré on automorphic functions. The fuchsian functions afford a means of uniform parametric representation of any algebraic curve, "certainly one of the most profound results obtained in analysis in the last fifty years."

In recent years Borel has made a penetrating study and comparison of the notion of *analyticity* in the sense of Weierstrass and of that of *monogeneity* in the sense of Cauchy, and has reached the conclusion that the latter is the essential one. The detailed investigations on which this conclusion is based are set forth in Borel's Leçons sur les Fonctions monogènes uniformes, 1917. The essential results, however, were announced at the Cambridge Congress in 1912.

In the hands of Picard the method of successive approximation has been a useful tool in the study of differential equations. Poincaré has introduced the general notion of asymptotic representation for the solutions of difference equations and of differential equations in the neighborhood of an irregular singular point. Poincaré has shown how linear algebraic differential equations may be integrated by means of thetafuchsian series. Painlevé has done fundamental work in the classification of non-linear differential equations with

fixed critical points. Poincaré has devoted several memoirs to the study of curves defined by differential equations and has made particularly notable contributions to the theory of the differential equations of physics. The work of Darboux on differential geometry is everywhere recognized as of leading importance. The same author has also created a new method of integrating partial differential equations; Goursat has devoted several papers to elucidating the questions suggested by this method.

The methods created by Hermite have given to the theory of numbers a new and broader horizon. The work of Jordan on the theory of forms has realized important progress in the general theory of algebraic forms of higher degree. Poincaré has left his mark on the theory of quadratic forms. Jordan has made a profound study of the epoch-making ideas of Galois. Halphen has extended the theory of twisted algebraic curves. Cartan has contributed to our knowledge of the structure of groups and to the determination of simple groups.

A considerable number of memoirs, of unequal value, have been devoted to the theory of assemblages. Picard believes that some of these are without interest to mathematics and characterizes their theory as a sort of *metamathematics* from which paradoxes are not absent. Of the important work in the theory of assemblages one may select for mention that of Borel on measure, that of Lebesgue on integration, and that of Baire on the classification of functions.

Sometimes mathematical researches are presented with an excessive formalism and symbolism, incapable of leading to a new fact and of being utilized in other researches than in those in which they were created. But when these last conditions are not realized one can think that there is no real scientific progress. In this respect it seems that French mathematics has wisely restrained itself within proper bounds and has not created a chapter of science which affords merely an exercise in logic. Whenever our science has a tendency to become too formal let us turn again to the work of the great creators, such as Fourier, Cauchy and Poincaré, and learn from them anew how to create other chapters in that body of mathematics which is not a strange and mysterious science, but an essential part of the structure of natural philosophy.

R. D. CARMICHAEL.

SHORTER NOTICE.

Elementary Mathematical Analysis. By JOHN WESLEY YOUNG and FRANK MILLETT MORGAN. New York, The Macmillan Company, 1917. 548 pp.

THIS book is intended to meet the need which has been felt by many during the last years, of a more unified course in freshman mathematics which, without unduly neglecting matters of technique, places more than the usual emphasis upon a genuine understanding of the mathematical concepts and methods. The book has its faults, as have most books, but, in the opinion of the reviewer, it is a fine contribution to the problems in hand and is really a class room text.

The central theme is the idea of functionality. This idea is skillfully kept before the reader's mind, though in the consideration of some topics, as that of trigonometric relations, it is rather lost sight of. This seems to be inherent in the situation and it is possibly unwise and somewhat artificial to insist that everything in such a course must associate itself immediately with the idea of functionality, important and far reaching as that idea is.

The material of the book is presented under five general headings.

In Part I (Introductory Conceptions) thirty-two pages are devoted to the general subject of functions and their representation and thirty-one pages to algebraic principles and their connection with geometry. Great pains is taken to familiarize the reader with the notion of functionality and with the geometrical representation of functions. Many concrete examples are exhibited in detail. Under the second heading is given an elaborate discussion of numbers and their geometrical representation, a statement of the fundamental laws of algebra and a review of elementary algebraic technique.

In Part II two hundred and twenty-nine pages are devoted to the elementary functions: the linear function; the quadratic function; the cubic function; the function x^n; the trigonometric functions with a special chapter on trigonometric relations; the logarithmic and exponential functions; numerical computation, including logarithmic solution of triangles, the slide rule, logarithmic paper; the implicit quadratic

functions. Under the last title equations of the second degree are classified according to their graphs. Thus the way is prepared for the conic section as the locus of a point which moves so that its distance from a fixed point is always equal to a constant times its distance from a fixed line.

Part III consists of 118 pages devoted to applications to geometry and contains a thorough treatment of the straight line, the circle, the conic sections, polar coordinates, and parametric equations.

In Part IV seventy-two pages are devoted to the consideration of such algebraic topics as permutations, combinations, probability, the binomial theorem, complex numbers, the general polynomial, the theory of equations and determinants.

Part V (forty-five pages) is a short treatment of the elements of solid analytic geometry.

The idea of slope as the limit of $\Delta y/\Delta x$ is introduced very early in Part II and is used wherever pertinent throughout the book. In addition to familiarizing the reader with this fundamental idea, the use of the slope facilitates the consideration of the various curves and enables the authors to make more of the subject of maxima and minima than is usual in a freshman course.

The trigonometric functions are defined at once for the general angle. The addition formulas are proved by means of the formulas for the rotation of axes, which have been previously considered. The chapter on logarithms is introduced by a careful, intuitive discussion of the exponential function, well calculated to give the reader a correct feeling as to the significance of exponents positive, negative, zero, rational, irrational.

The avowed purpose of the authors to place "more emphasis on insight and understanding of fundamental conceptions and modes of thought" is well carried out both in the text and in the problem lists. In the text are numerous questions tending to make the student read more thoughtfully and the problem lists contain many exercises of various degrees of difficulty which focus on the fundamental principles under consideration rather than on manipulation or technique. However there is a sufficient number of problems of all kinds and the lists are properly graded.

In the opinion of the reviewer, Part I is too long and (especially chapter two) contains too much discussion not well

chosen to introduce the subject to students during their first days in college. Chapter three, on the linear function, and certain parts of the sequel likewise seem to be somewhat verbose and indirect.

The book is well adapted for use by students who enter with trigonometry as well as by those who enter without trigonometry. The reviewer has felt that the program outlined in the preface is somewhat too ambitious and still believes that the average class will have some difficulty in covering the ground indicated. This emphasizes the need for greater brevity and directness in certain parts of the text. However by careful selection the needs of almost any class can be satisfied.

In the preface the authors announce the book as somewhat of an experiment and suggest a possible revision. It is too much to expect that, in these troublous times, such a revision may appear soon but it is to be hoped that it may not be delayed too long. In view of the experience of the authors themselves with the present book, a revised edition would surely be a finished product. The reviewer is, with the authors, in doubt as to the advisability of including more calculus in the text. Probably this can be determined only by experiment.

The book is attractive in print and in appearance and the figures are good. There are minor errors but the reviewer has made no effort to list these.

Any teacher who is interested in the matter of a more unified course for the freshman year would do well to give this book a careful examination. Any teacher of freshman mathematics who is interested in securing, on the part of his students, a better understanding and appreciation of the principles and methods of mathematics will find much here that is helpful whether or not he is interested in this particular type of course.

A. D. PITCHER.

NOTES.

THE concluding (October) number of volume 19 of the *Transactions of the American Mathematical Society* contains the following papers: "Spiral minimal surfaces," by J. K. WHITTEMORE; "On the group of isomorphisms of a certain extension of an abelian group," by L. C. MATHEWSON; "Concerning the zeros of the solutions of certain differential equations," by W. B. FITE; "Differentiation with respect to a function of limited variation," by P. J. DANIELL; "Linear integro-differential equations with a boundary condition," by M. T. HU; "On scalar and vector covariants of linear algebras," by OLIVE C. HAZLETT.

AT the meeting of the Edinburgh mathematical society on November 8, the following paper was read: By J. E. A. STEGGAL, "Notes on the homographic transformation of the cubic and biquadratic."

THERE has recently been issued by the Bureau of Education at Washington a Bulletin on the training of teachers of mathematics for secondary schools of the countries represented in the International commission on the teaching of mathematics. This Bulletin has been prepared by Professor R. C. ARCHIBALD, of Brown University. It is a work of nearly three hundred pages, giving in great detail the requirements set by the various governments for a teacher of secondary mathematics. The Bureau of Education has a limited number of copies for free distribution to those who are particularly interested in the work, and copies can be obtained from the Superintendent of Documents, Government Printing Office at Washington, D. C., at 30 cents each.

HARVARD UNIVERSITY. The winter term of the graduate school of arts and sciences will begin January 2 and close March 20. The spring term will begin March 28 and close with Commencement Day, June 19. It is expected that a summer term of eleven weeks will be arranged, so that a student beginning his work in the winter may complete a full year of residence by September. Those to whom fellowships or scholarships

had been assigned for the academic year 1918–19, but who, because of national service, resigned these, should apply promptly for reappointment. A certain number of fellowships and scholarships are now vacant, for which applications will be received.

PROFESSOR C. CARATHÉODORY, of the University of Göttingen, has been appointed professor of mathematics at the University of Berlin, as successor to the late Professor G. FROBENIUS.

PROFESSOR G. C. EVANS, of the Rice Institute, has been commissioned captain of ordnance in the United States Army and sent to France on a special mission.

LIEUTENANT T. R. HOLLCROFT has been released from the teaching staff of the school of artillery at Camp Taylor, and has resumed the duties of his professorship of mathematics at Wells College. During his absence the position was filled by Mrs. Hollcroft.

DR. T. M. SIMPSON, of the University of Wisconsin, has been made professor and head of the department of mathematics at the University of Florida, as successor to the late Professor H. G. KEPPEL.

PROFESSOR J. W. YOUNG, of Dartmouth College, has been made director of the instruction in mathematics given under the National war work council of the Young Men's Christian Association. During his absence and that of Professors HASKINS and BILL, also engaged in war work, the administration of the department of mathematics is in the hands of Professor F. M. MORGAN.

DR. S. D. ZELDIN, of the College of Hawaii, has been appointed professor of mathematics at Olivet College.

PROFESSOR W. O. MENDENHALL, of Earlham College, has been elected to the presidency of the Friends University, Wichita, Kansas.

MR. C. A. HUTCHINSON has been appointed instructor in mathematics at the University of Colorado.

DR. J. E. MCATEE, of the University of Illinois, died December 2, 1918.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BELL (R. J. T.). See FROST (P.).

BENEDICT (H. Y.). See KARPINSKI (L. C.).

CALHOUN (J. W.). See KARPINSKI (L. C.).

COTES (R.). See KOWALEWSKI (A.).

CULLIS (C. E.). Matrices and determinoids. Volume 2. Cambridge University Press, 1918. 24+555 pp. 42 s.

DONADT (A.). See LÜBSEN (H. B.).

FISCHER (L. A.). See HUNTINGTON (E. V.).

FROST (P.). An elementary treatise on curve tracing. 4th edition, by R. J. T. Bell. London, Macmillan, 1918. 16+210 pp. 12s. 6d.

GAUSS (C. F.). See KOWALEWSKI (A.).

GIGNANO (E.). Essays in scientific synthesis. Translated by W. J. Greenstreet. Chicago, Open Court, 1918. 235 pp. $2.00

GREENSTREET (W. J.). See GIGNANO (E.).

JACOBI (C. G. J.). See KOWALEWSKI (A.).

KARPINSKI (L. C.), BENEDICT (H. Y.) and CALHOUN (J. W.). Unified mathematics. Boston, Heath, 1918. 8 + 522 pp. $2.80

KOWALEWSKI (A.). Newton, Cotes, Gauss, Jacobi. Vier grundlegende Abhandlungen über Interpolation und genäherte Quadratur. Herausgegeben von A. Kowalewski. Leipzig, Veit, 1917. Geh. 104 pp. M. 4.80

LÜBSEN (H. B.). Einleitung in die Infinitesimalrechnung (Differential- und Integralrechnung) zum Selbstunterricht. Mit Rücksicht auf das Notwendigste und Wichtigste. Neu bearbeitet von A. Donadt. 9te Auflage. Leipzig, F. Brandstetter, 1916. 8vo. 7+440 pp. M. 8.00

NEWTON (I.). See KOWALEWSKI (A.).

PIRA (K.). Om idesystem och talsystem (Smärre Skrifter utg. af Boströmsförbundet, 38–40). Del 1–3. Stockholm, A. B. A. Johansen, 1913–14. 8vo. 126 + 108 + 184 pp. Kr.1.50+1.25+2.00

RANSOM (W. R.). Freshman mathematics brief, simple, practical for the Freshman college classes. New York, Longmans, 1918. 320 pp. $1.50

TOLEDO (L. O. de). Elementos de aritmética universal. Tomo 2: Coordinatoria, determinantes, algoritmos ilimitados. Madrid, Librería de la Vda é Hijos de Murillo, 1916. 8vo. 421 pp. Pes. 14.00

VOIGT (A.). Die Teilbarkeit der Potenzsummen und die Lösung des Fermat'schen Problems. Eine zahlenteoretische Untersuchung. Frankfurt a.M., M. Diesterweg, 1916. 8vo. 12+105 pp. M. 2.00

II. ELEMENTARY MATHEMATICS.

ALZÁA (F. de). Nociones de geometría plana. Buenos Aires, A. Garcia Santos, 1917.

GOMEZ RUIZ (A.). Tratado de trigonometria plana, circular y hiperbolica. (Memorias de la Real Academia de Ciencias, Madrid.) Madrid, De Fortanet, 1917. 4to. 318 pp.

KING (W. J.). A condensed geometry review with answers. Annapolis, Md., C. G. Feldmeyer, 1918. 8vo. 35 pp.

MÜLLER (H.). Die Mathematik auf den Gymnasien und Realschulen. Ausgabe A: Für Gymnasien. 2ter Teil: Die Oberstufe. 4te, umgearbeitete Auflage. Leipzig, Teubner, 1917. 14+360 pp.

TOLEDO (L. O. de). Tratado de álgebra. Tomo 1: Parte elemental. Seg. edition. Madrid, Libreria General de Victoriano Suárez, 1914. 8vo. 11+362 pp. Pes. 10.00

—— Tratado de trigonometria rectilínea y esférica. Seg. edition. Madrid, Libreria General de Victoriano Suárez, 1914. 8vo. 285 pp. Pes. 8.00

III. APPLIED MATHEMATICS.

ARRHENIUS (S.). The destinies of the stars, by S. Arrhenius. Authorized translation from the Swedish by J. E. Fries. New York, Putnam, 1918. $1.50

ATTWOOD (E. L.). Textbook of theoretical naval architecture. 9th impression. New York, Longmans, 1918. 8vo. 12+494 pp. $3.50

BECQ (A.). Applications industrielles de l'électricité. Transmission, distribution, et utilisation de l'énergie électrique. Paris, Librairie de l'Ecole spéciale des Travaux publics, 1917. 8vo. 231 pp. Fr. 12.00

DEVILLERS (R.). Le moteur à explosions. Préface de C. Faroux. Paris, Dunod et Pinat, 1917. 8vo. 8+422 pp. Fr. 8.25

DONNER (A.). See FESTSKRIFT.

DORGEOT (E.). La mécanique appliquée, théorique, numérique et graphique. Paris, Dunod et Pinat, 1918. 4to. 613 pp. Fr. 39.00

FAROUX (C.). Dee DEVILLERS (R.).

FESTSKRIFT tillegnad Anders Donner på hans sextioårsdag den 5 nov. 1914 af forne elever. Helsingfors, J. Simellii Arfvingars Boktryckeria-Ktiebolog, 1915. 4to. 410 pp.

FRIES (J. E.). See ARRHENIUS (S.).

FÜRST (A.) UND MOSZKOWSKI (A.). Das Buch der 1000 Wunder. München, A. Lange, 1917. Geh. M. 6.00

HOLMS (A. C.). Practical shipbuilding. A treatise on the structural design and building of modern steam vessels. Volume 1: Text. Volume 2: Diagrams and illustrations. 6th impression. New York, Longmans, 1918. $20.00

HUNTINGTON (E. V.). Handbook of mathematics for engineers by E. V. Huntington with tables of weights and measures by L. A. Fischer. New York, McGraw-Hill, 1918. 8vo. 191 pp. $1.50

HUTCHINSON (R. W.). Advanced textbook of magnetism and electricity. London, University Tutorial Press, 1917.

KAYE (G. W. C.). Tables of physical and chemical constants and some mathematical functions. 3d edition. New York, Longmans, 1918. 8vo. 8+153 pp. $2.25

—— X-rays. London, Longmans, 1917.

KAUFMANN (A.). Theorie und Methoden der Statistik. Ein Lehr- und Lesebuch für Studierende und Praktiker. Tübingen, Mohr, 1913. 8vo. 12+540 pp.

LOW (D. A.). A pocketbook for mechanical engineers. 7th impression. New York, Longmans, 1918. 8vo. 8+740 pp. $3.00

MAUNDER (E. W.). The stars and how to identify them. London, Kelly, 1918. 63 pp.

MOSZKOWSKI (A.). See FÜRST (A.).

PLUMMER (H. C.). An introductory treatise on dynamical astronomy. Cambridge, University Press, 1918. Royal 8vo. 18s.

SCHOUTEN (W. J. A.). On the determination of the principal laws of statistical astronomy. Amsterdam, W. Kirchner, 1918. 128 pp.

SMITH (H. E.). Mechanics; prepared for the use of midshipmen and adopted as the textbook at the Naval Academy. Annapolis, U. S. Naval Institute, 1918. 8vo. 11+269 pp. $3.60

STEINMETZ (C. P.). Engineering mathematics; a series of lectures delivered at Union College. 3d edition, revised and enlarged. New York, McGraw-Hill, 1917. 8vo. 19+321 pp. $3.00

STORMER (C.). Sur un problème relatif au mouvement des corpuscules éléctriques dans l'espace cosmique. (Utgit for Fridtjof Nansens Fond.) Kristiania, J. Dybwad, 1916. 8vo. 60+91 pp.

SVENSEN (C. L.). Essentials of drafting; text and problem book for apprentice, trade and evening technical schools. New York, Van Nostrand, 1918. 8vo. 7–14+184+31 pp. $1.50